

Lecture 04: Optimum Design Problem Formulation



Helicopter Rotor Design

Space Station Power System

Shot peen forming of wing skins

Aerospika Nozzle

Multidisciplinary wing platform design

3-D Fighter Aerodynamics

Engine Nozzle Performance

777 Engine Duct Seals

Machining, riveting, and drilling database

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Outline

- Introduction
- Five-step formulation procedure
- General mathematical model for optimum design
- Classification of optimization problems

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Introduction

- **Proper definition and formulation** of a problem take roughly 50 percent of the total effort needed to solve it.
- The optimum solution will be only **as good as the formulation**.

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Five-step Formulation Procedure

- Step 1: Project/problem description
- Step 2: Data and information collection
- Step 3: Definition of design variables
- Step 4: Optimization criterion
- Step 5: Formulation of constraints

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Step 1: Project/Problem Description

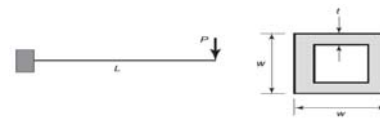
- The statement describes the overall objectives of the project and the requirements to be met.
- This is also called the statement of work.

Example 1: Design of a Cantilever Beam—Problem Description

Cantilever beams are used in many practical applications in civil, mechanical, and aerospace engineering. To illustrate the step of problem description, we consider the design of a hollow square-cross-section cantilever beam to support a load of 20 kN at its end. The beam, made of steel, is 2 m long, as shown in Figure 2.1. The failure conditions for the beam are as follows: (1) the material should not fail under the action of the load, and (2) the deflection of the free end should be no more than 1 cm. The width-to-thickness ratio for the beam should be no more than 8. A minimum-mass beam is desired. The width and thickness of the beam must be within the following limits:

$$60 \leq \text{width} \leq 300 \text{ mm} \quad (a)$$

$$10 \leq \text{thickness} \leq 40 \text{ mm} \quad (b)$$



Step 2: Data and Information Collection

EXAMPLE 2.2 DATA AND INFORMATION COLLECTION FOR A CANTILEVER BEAM

The information needed for the cantilever beam design problem of Example 2.1 includes expressions for bending and shear stresses, and the expression for the deflection of the free end. The notation and data for this purpose are defined in the table that follows.

Useful expressions for the beam are

$$A = w^2 - (w - 2t)^2 = 4t(w - t), \text{ mm}^2 \quad (c)$$

$$I = \frac{8}{3}wt^3 + \frac{2}{3}w^3t - 2wt^2t - \frac{4}{3}t^4, \text{ mm}^4 \quad (d)$$

$$Q = \frac{3}{4}wt^2t - \frac{3}{2}wt^2 + t^3, \text{ mm}^3 \quad (e)$$

$$M = PL, \text{ N} \cdot \text{mm} \quad (f)$$

$$V = P, \text{ N} \quad (g)$$

$$\sigma = \frac{Mw}{2I}, \text{ N} \cdot \text{mm}^{-2} \quad (h)$$

$$\tau = \frac{VQ}{2It}, \text{ N} \cdot \text{mm}^{-2} \quad (i)$$

$$q = \frac{PL^3}{3EI}, \text{ mm} \quad (j)$$

Step 2: Data and Information Collection

Notation	Data
A	cross-sectional area, mm ²
E	modulus of elasticity, 21 × 10 ⁶ N · mm ⁻²
G	shear modulus, 8 × 10 ⁴ N · mm ⁻²
I	moment of inertia, mm ⁴
L	length of the member, 2000 mm
M	bending moment, N · mm
P	load at the free end, 20,000 N
Q	moment about the neutral axis of the area above the neutral axis, mm ³
q	vertical deflection of the free end, mm
q _a	allowable vertical deflection of the free end, 10 mm
V	shear force, N
w	width (depth) of the section, mm
t	wall thickness, mm
σ	bending stress, N · mm ⁻²
σ _a	allowable bending stress, 165 N · mm ⁻²
τ	shear stress, N · mm ⁻²
τ _a	allowable shear stress, 90 N · mm ⁻²

Step 3: Definition of Design Variables



- The design variables should be **independent of each other** as far as possible.
- The number of independent design variables gives the **design degrees of freedom** for the problem.
- A **minimum number of design variables** required to properly formulate a design optimization problem must exist.
- As many independent parameters as possible should be designated as design variables at the problem formulation phase → later on, some of the variables can be assigned fixed values.
- A numerical value should be given to each identified design variable to determine if a trial design of the system is specified.

Step 3: Definition of Design Variables



EXAMPLE 2.3 DESIGN VARIABLES FOR A CANTILEVER BEAM

Only dimensions of the cross-section are identified as design variables for the *cantilever beam design problem* of Example 2.1; all other parameters are specified:

w = width (depth) of the section, mm
 t = wall thickness, mm

Step 4: Optimization Criterion



- It must be a function of the design variable vector \mathbf{x} .
- Such a criterion is usually called an **objective function** for the optimum design problem, and it needs to be **maximized** or **minimized** depending on problem requirements.

EXAMPLE 2.4 OPTIMIZATION CRITERION FOR A CANTILEVER BEAM

For the *design problem in Example 2.1*, the objective is to design a minimum-mass cantilever beam. Since the mass is proportional to the cross-sectional area of the beam, the objective function for the problem is taken as the cross-sectional area:

$$f(w, t) = A = 4t(w - t), \text{ mm}^2 \quad (k)$$

Step 5: Formulation of Constraints



- All restrictions placed on the design are collectively called **constraints**.
- Most realistic systems must be designed and fabricated with the given **resources** and must **meet performance requirements**.
- A meaningful constraint must be a function of **at least one design variable**.
- Concepts and terms related to constraints:
 - Linear and Nonlinear Constraints
 - Feasible Design
 - Equality and Inequality Constraints
 - Implicit Constraints

Step 5: Formulation of Constraints

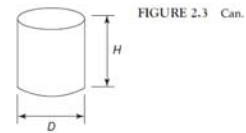
EXAMPLE 2.5 CONSTRAINTS FOR A CANTILEVER BEAM

Using various expressions given in Eqs. (c) through (p), we formulate the constraints for the cantilever beam design problem from Example 2.1 as follows:

Bending stress constraint: $\sigma \leq \sigma_a$	$\frac{PLw}{2I} - \sigma_a \leq 0$	(l)
Shear stress constraint: $\tau \leq \tau_a$	$\frac{PQ}{2I} - \tau_a \leq 0$	(m)
Deflection constraint: $q \leq q_a$	$\frac{PL^3}{3EI} - q_a \leq 0$	(n)
Width-thickness restriction: $\bar{y} \leq 8$	$w - 8t \leq 0$	(o)
Dimension restrictions	$60 - w \leq 0, \text{ mm}; w - 300 \leq 0, \text{ mm}$	(p)
	$3 - t \leq 0, \text{ mm}; t - 15 \leq 0, \text{ mm}$	(q)

Design of a Can

STEP 1: PROJECT/PROBLEM DESCRIPTION The purpose of this project is to design a can, shown in Figure 2.3, to hold at least 400 ml of liquid (1 ml = 1 cm³), as well as to meet other design requirements. The cans will be produced in the billions, so it is desirable to minimize their manufacturing costs. Since cost can be directly related to the surface area of the sheet metal used, it is reasonable to minimize the amount of sheet metal required. Fabrication, handling, aesthetics, and shipping considerations impose the following restrictions on the size of the can: The diameter should be no more than 8 cm and no less than 3.5 cm, whereas the height should be no more than 18 cm and no less than 8 cm.



Design of a Can

STEP 2: DATA AND INFORMATION COLLECTION Data for the problem are given in the project statement.

STEP 3: DEFINITION OF DESIGN VARIABLES The two design variables are defined as
 D = diameter of the can, cm
 H = height of the can, cm

STEP 4: OPTIMIZATION CRITERION The design objective is to minimize the total surface area S of the sheet metal for the three parts of the cylindrical can: the surface area of the cylinder (circumference × height) and the surface area of the two ends. Therefore, the optimization criterion, or cost function (the total area of sheet metal), is given as

$$S = \pi DH + 2\left(\frac{\pi}{4}D^2\right), \text{ cm}^2 \quad (a)$$

Design of a Can

STEP 5: FORMULATION OF CONSTRAINTS The first constraint is that the can must hold at least 400 cm³ of fluid, which is written as

$$\frac{\pi}{4}D^2H \geq 400, \text{ cm}^3 \quad (b)$$

If it had been stated that “the can must hold 400 ml of fluid,” then the preceding volume constraint would be an equality. The other constraints on the size of the can are

$$\begin{aligned} 3.5 \leq D \leq 8, \text{ cm} \\ 8 \leq H \leq 18, \text{ cm} \end{aligned} \quad (c)$$

Side constraints, technological constraints, simple bounds, sizing constraints, and upper and lower limits on the design variables

Design of a Cabinet

STEP 1: PROJECT/PROBLEM DESCRIPTION A cabinet is assembled from components C_1 , C_2 , and C_3 . Each cabinet requires 8 C_1 , 5 C_2 , and 15 C_3 components. The assembly of C_1 requires either 5 bolts or 5 rivets, whereas C_2 requires 6 bolts or 6 rivets, and C_3 requires 3 bolts or 3 rivets. The cost of installing a bolt, including the cost of the bolt itself, is \$0.70 for C_1 , \$1.00 for C_2 , and \$0.60 for C_3 . Similarly, riveting costs are \$0.60 for C_1 , \$0.80 for C_2 , and \$1.00 for C_3 . Bolting and riveting capacities per day are 6000 and 8000, respectively. To minimize the cost for the 100 cabinets that must be assembled each day, we wish to determine the number of components to be bolted and riveted (after Siddall, 1972).

STEP 2: DATA AND INFORMATION COLLECTION All data for the problem are given in the project statement. This problem can be formulated in several different ways depending on the assumptions made and the definition of the design variables. Three formulations are presented, and for each one, the design variables are identified and expressions for the cost and constraint functions are derived; that is, steps 3 through 5 are presented.

Formulation 1 for Cabinet Design

STEP 3: DEFINITION OF DESIGN VARIABLES In the first formulation, the following design variables are identified for 100 cabinets:

x_1 = number of C_1 to be bolted for 100 cabinets
 x_2 = number of C_1 to be riveted for 100 cabinets
 x_3 = number of C_2 to be bolted for 100 cabinets
 x_4 = number of C_2 to be riveted for 100 cabinets
 x_5 = number of C_3 to be bolted for 100 cabinets
 x_6 = number of C_3 to be riveted for 100 cabinets

STEP 4: OPTIMIZATION CRITERION The design objective is to minimize the total cost of cabinet fabrication, which is obtained from the specified costs for bolting and riveting each component:

$$\begin{aligned} \text{Cost} &= 0.70(5)x_1 + 0.60(5)x_2 + 1.00(6)x_3 + 0.80(6)x_4 + 0.60(3)x_5 + 1.00(3)x_6 \\ &= 3.5x_1 + 3.0x_2 + 6.0x_3 + 4.8x_4 + 1.8x_5 + 3.0x_6 \end{aligned} \quad (a)$$

Formulation 1 for Cabinet Design

STEP 5: FORMULATION OF CONSTRAINTS The constraints for the problem consist of riveting and bolting capacities and the number of cabinets fabricated each day. Since 100 cabinets must be fabricated, the required numbers of C_1 , C_2 , and C_3 are given in the following constraints:

$$\begin{aligned} x_1 + x_2 &= 8 \times 100 \quad (\text{number of } C_1 \text{ needed}) \\ x_3 + x_4 &= 5 \times 100 \quad (\text{number of } C_2 \text{ needed}) \\ x_5 + x_6 &= 15 \times 100 \quad (\text{number of } C_3 \text{ needed}) \end{aligned} \quad (b)$$

Bolting and riveting capacities must not be exceeded. Thus,

$$\begin{aligned} 5x_1 + 6x_3 + 3x_5 &\leq 6000 \quad (\text{bolting capacity}) \\ 5x_2 + 6x_4 + 3x_6 &\leq 8000 \quad (\text{riveting capacity}) \end{aligned} \quad (c)$$

Finally, all design variables must be non-negative to find a meaningful solution:

$$x_i \geq 0; \quad i = 1 \text{ to } 6 \quad (d)$$

Formulation 2 for Cabinet Design

STEP 3: DEFINITION OF DESIGN VARIABLES If we relax the constraint that each component must be bolted or riveted, then the following design variables can be defined:

x_1 = total number of bolts required for all C_1
 x_2 = total number of bolts required for all C_2
 x_3 = total number of bolts required for all C_3
 x_4 = total number of rivets required for all C_1
 x_5 = total number of rivets required for all C_2
 x_6 = total number of rivets required for all C_3

STEP 4: OPTIMIZATION CRITERION The objective is still to minimize the total cost of fabricating 100 cabinets, given as

$$\text{Cost} = 0.70x_1 + 1.00x_2 + 0.60x_3 + 0.60x_4 + 0.80x_5 + 1.00x_6 \quad (e)$$

Formulation 2 for Cabinet Design

STEP 5: FORMULATION OF CONSTRAINTS Since 100 cabinets must be built every day, it will be necessary to have 800 C_1 , 500 C_2 , and 1500 C_3 components. The total number of bolts and rivets needed for all C_1 , C_2 , and C_3 components is indicated by the following equality constraints:

$$\begin{aligned} x_1 + x_4 &= 5 \times 800 && \text{(bolts and rivets needed for } C_1) \\ x_2 + x_5 &= 6 \times 500 && \text{(bolts and rivets needed for } C_2) \\ x_3 + x_6 &= 3 \times 1500 && \text{(bolts and rivets needed for } C_3) \end{aligned} \quad (f)$$

Constraints on capacity for bolting and riveting are

$$\begin{aligned} x_1 + x_2 + x_3 &\leq 6000 && \text{(bolting capacity)} \\ x_4 + x_5 + x_6 &\leq 8000 && \text{(riveting capacity)} \end{aligned} \quad (g)$$

Finally, all design variables must be non-negative:

$$x_i \geq 0; \quad i = 1 \text{ to } 6 \quad (h)$$

Formulation 3 for Cabinet Design

STEP 3: DEFINITION OF DESIGN VARIABLES Another formulation of the problem is possible if we require that all cabinets be identical. The following design variables can be identified:

$$\begin{aligned} x_1 &= \text{number of } C_1 \text{ to be bolted on one cabinet} \\ x_2 &= \text{number of } C_1 \text{ to be riveted on one cabinet} \\ x_3 &= \text{number of } C_2 \text{ to be bolted on one cabinet} \\ x_4 &= \text{number of } C_2 \text{ to be riveted on one cabinet} \\ x_5 &= \text{number of } C_3 \text{ to be bolted on one cabinet} \\ x_6 &= \text{number of } C_3 \text{ to be riveted on one cabinet} \end{aligned}$$

STEP 4: OPTIMIZATION CRITERION With these design variables, the cost of fabricating 100 cabinets each day is given as

$$\begin{aligned} \text{Cost} &= 100[0.7(5)x_1 + 0.60(5)x_2 + 1.00(6)x_3 + 0.80(6)x_4 + 0.60(3)x_5 + 1.00(3)x_6] \\ &= 350x_1 + 300x_2 + 600x_3 + 480x_4 + 180x_5 + 300x_6 \end{aligned} \quad (i)$$

Formulation 3 for Cabinet Design

STEP 5: FORMULATION OF CONSTRAINTS Since each cabinet needs 8 C_1 , 5 C_2 , and 15 C_3 components, the following equality constraints can be identified:

$$\begin{aligned} x_1 + x_2 &= 8 && \text{(number of } C_1 \text{ needed)} \\ x_3 + x_4 &= 5 && \text{(number of } C_2 \text{ needed)} \\ x_5 + x_6 &= 15 && \text{(number of } C_3 \text{ needed)} \end{aligned} \quad (j)$$

Constraints on the capacity to rivet and bolt are expressed as the following inequalities:

$$\begin{aligned} (5x_1 + 6x_3 + 3x_5)100 &\leq 6000 && \text{(bolting capacity)} \\ (5x_2 + 6x_4 + 3x_6)100 &\leq 8000 && \text{(riveting capacity)} \end{aligned} \quad (k)$$

Finally, all design variables must be non-negative:

$$x_i \geq 0; \quad i = 1 \text{ to } 6 \quad (l)$$

A General Mathematical Model for Optimum Design

Standard Design Optimization Model

Find an n -vector $x = (x_1, x_2, \dots, x_n)$ of design variables to

Minimize a cost function:

$$f(x) = f(x_1, x_2, \dots, x_n)$$

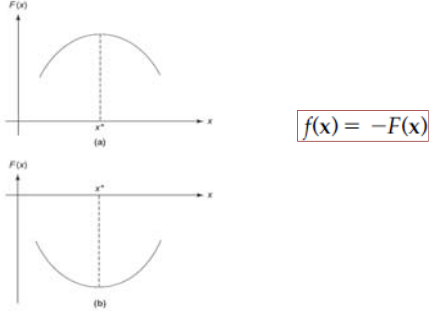
subject to the p equality constraints:

$$h_j(x) = h_j(x_1, x_2, \dots, x_n) = 0; \quad j = 1 \text{ to } p$$

and the m inequality constraints:

$$g_i(x) = g_i(x_1, x_2, \dots, x_n) \leq 0; \quad i = 1 \text{ to } m$$

Maximization Problem Treatment



(a)

(b)

$f(x) = -F(x)$

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Treatment of “Greater Than Type” Constraints

$$G_j(x) \geq 0 \rightarrow g_j(x) = -G_j(x) \leq 0.$$

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Important Observations about the Standard Model

1. Dependence of functions on design variables
2. Number of equality constraints
3. Number of inequality constraints
4. Unconstrained problems
5. Linear programming problems
6. Scaling of problem functions

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Feasible Set

$$S = \{x \mid h_i(x) = 0, j = 1 \text{ to } p; g_i(x) \leq 0, i = 1 \text{ to } m\}$$

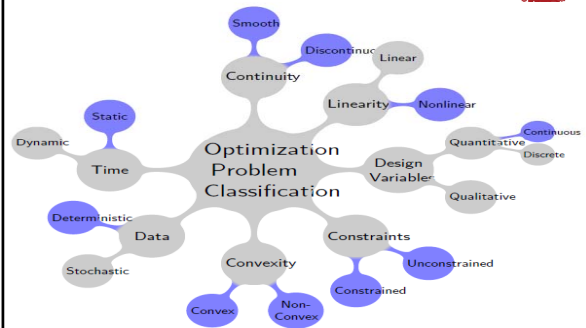
- Feasible region usually **shrinks** when more constraints are **added** to the design model and expands when some constraints are deleted.

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Active/Inactive/Violated Constraints

- Constraints → active, tight, binding, inactive, or violated.
- An inequality constraint $g_j(x) \leq 0$ is said to be **active** at a design point x^* if it is satisfied at equality (i.e., $g_j(x^*) = 0$) → also called tight or binding.
- All equality constraints are active for all **feasible designs**.
- An inequality constraint $g_j(x) \leq 0$ is said to be **inactive** at a design point x^* if it is strictly satisfied (i.e., $g_j(x^*) < 0$).
- It is said to be **violated** at a design point x^* if its value is positive (i.e., $g_j(x^*) > 0$).
- An equality constraint $h_i(x)=0$ is **violated** at a design point x^* if $h_i(x^*)$ is not identically zero.
- An equality constraint is either **active** or **violated** at a given design point.

Classification of Optimization Problems



Assignment-1

- Arora Chapter 2:
 - Problems 2.5, 2.7, 2.11, 2.16, 2.17.